



The Dynamics of

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}$$

Facts and Conjectures

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Abstract—We investigate the global character of solutions of the equation in the title with non-negative parameters and nonnegative initial conditions. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

We investigate the global character of solutions of the second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots, \quad (1)$$

with nonnegative parameters α, β, A, B, C , and nonnegative initial conditions x_{-1}, x_0 .

Other nonlinear, second-order, rational difference equations were investigated in [1–11].

The study of rational difference equations of order greater than one is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one. The techniques and results about these equations are also useful in analyzing the equations in the mathematical models of various biological systems and other applications.

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$. Let $\bar{x} \in I$ be an equilibrium point of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots; \quad (2)$$

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that is,

$$\bar{x} = f(\bar{x}, \bar{x}).$$

DEFINITION 1.

- (i) The equilibrium \bar{x} of equation (2) is called *locally stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x_0, x_{-1} \in I$ with $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \delta$, then

$$|x_n - \bar{x}| < \varepsilon, \quad \text{for all } n \geq -1.$$

- (ii) The equilibrium \bar{x} of equation (2) is called *locally asymptotically stable* if it is locally stable, and if there exists $\gamma > 0$ such that $x_0, x_{-1} \in I$ with $|x_0 - \bar{x}| + |x_{-1} - \bar{x}| < \gamma$, then

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

- (iii) The equilibrium \bar{x} of equation (2) is called a *global attractor* if for every $x_0, x_{-1} \in I$, we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

- (iv) The equilibrium \bar{x} of equation (2) is called *globally asymptotically stable* if it is locally stable and a global attractor.

- (v) The equilibrium \bar{x} of equation (2) is called *unstable* if it is not stable.

Let

$$s = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) \quad \text{and} \quad t = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium \bar{x} of equation (2). Then the equation

$$y_{n+1} = sy_n + ty_{n-1}, \quad n = 0, 1, \dots, \quad (3)$$

is called the *linearized equation* associated with equation (2) about the equilibrium point \bar{x} .

THEOREM A. LINEARIZED STABILITY.

- (a) If both roots of the quadratic equation

$$\lambda^2 - s\lambda - t = 0 \quad (4)$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium \bar{x} of equation (2) is locally asymptotically stable.

- (b) If at least one of the roots of equation (4) has absolute value greater than one, then the equilibrium \bar{x} of equation (2) is unstable.

- (c) A necessary and sufficient condition for both roots of equation (4) to lie in the open unit disk $|\lambda| < 1$ is

$$|s| < 1 - t < 2. \quad (5)$$

In this case, the locally asymptotically stable equilibrium \bar{x} is also called a *sink*.

- (d) A necessary and sufficient condition for both roots of equation (4) to have absolute value greater than one is

$$|t| > 1 \quad \text{and} \quad |s| < |1 - t|.$$

In this case, \bar{x} is a *repeller*.

- (e) A necessary and sufficient condition for one root of equation (4) to have absolute value greater than one and for the other to have absolute value less than one is

$$s^2 + 4t > 0 \quad \text{and} \quad |s| > |1 - t|.$$

In this case, the unstable equilibrium \bar{x} is called a *saddle point*.

We now give the definitions of positive and negative semicycles of a solution of equation (2) relative to an equilibrium point \bar{x} . We believe that a semicycle analysis of the solutions of a scalar difference equation is a powerful tool for a detailed understanding of the entire character of solutions and often leads to proofs of their long term behavior.

A *positive semicycle* of a solution $\{x_n\}$ of equation (2) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to the equilibrium \bar{x} , with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either } l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

A *negative semicycle* of a solution $\{x_n\}$ of equation (2) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than the equilibrium \bar{x} , with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either } l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

The next three results are general convergence theorems for equation (2) which we will use in the sequel.

THEOREM B. (See [10].) Let $[a, b]$ be an interval of real numbers and assume that

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is a continuous function satisfying the following properties:

- (a) $f(x, y)$ is nondecreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is nonincreasing in $y \in [a, b]$ for each $x \in [a, b]$;
- (b) if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, M) = m \quad \text{and} \quad f(M, m) = M,$$

then $m = M$.

Then equation (2) has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of equation (2) converges to \bar{x} .

THEOREM C. (See [2].) Let $[a, b]$ be a interval of real numbers and assume that

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is a continuous function satisfying the following properties:

- (a) $f(x, y)$ is nonincreasing in each of its arguments;
- (b) if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, m) = M \quad \text{and} \quad f(M, M) = m,$$

then $m = M$.

Then equation (2) has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of equation (2) converges to \bar{x} .

THEOREM D. (See [5, p. 27].) Assume

- (i) $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$;
- (ii) $f(x, y)$ is nonincreasing in x and decreasing in y ;
- (iii) $xf(x, x)$ is increasing in x ;
- (iv) the equation

$$x_{n+1} = x_n f(x_n, x_{n-1}), \quad n = 0, 1, \dots, \quad (6)$$

has a unique positive equilibrium \bar{x} .

Then \bar{x} is globally asymptotically stable.

The next result is known as the *stability trichotomy* result.

THEOREM E. (See [12].) Consider the difference equation (2) where

$$f \in C^1[[0, \infty) \times [0, \infty), [0, \infty))$$

and such that

$$x \left| \frac{\partial f}{\partial x} \right| + y \left| \frac{\partial f}{\partial y} \right| < f(x, y), \quad \text{for all } x, y \in (0, \infty). \quad (7)$$

Then equation (2) has *stability trichotomy*; that is, exactly one of the following three cases holds for all solutions of equation (2):

- (i) $\lim_{n \rightarrow \infty} x_n = \infty$ for all $(x_{-1}, x_0) \neq (0, 0)$;
- (ii) $\lim_{n \rightarrow \infty} x_n = 0$ for all initial points and 0 is the only equilibrium point of equation (2);
- (iii) $\lim_{n \rightarrow \infty} x_n = \bar{x} \in (0, \infty)$ for all $(x_{-1}, x_0) \neq (0, 0)$ and \bar{x} is the only positive equilibrium of equation (2).

2. THE SPECIAL CASES WHERE $\alpha\beta ABC = 0$

If we allow the parameters α, β, A, B, C to be nonnegative, then equation (1) contains, as special cases, 21 difference equations with positive parameters. Of these, 12 equations are trivial, linear, reducible to linear, or of the Riccati type

$$y_{n+1} = \frac{ay_n + b}{cy_n + d}, \quad n = 0, 1, \dots,$$

with nonnegative parameters a, b, c, d , which itself is reducible to a linear equation by a well-known change of variables. See [13–16].

The remaining nine equations are the following, where we now assume that all their parameters are positive.

$$(i) \quad x_{n+1} = \frac{\alpha}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (8)$$

$$(ii) \quad x_{n+1} = \frac{\beta x_n}{A + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (9)$$

$$(iii) \quad x_{n+1} = \frac{\beta x_n}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (10)$$

$$(iv) \quad x_{n+1} = \frac{\alpha + \beta x_n}{Cx_{n-1}}, \quad n = 0, 1, \dots \quad (11)$$

$$(v) \quad x_{n+1} = \frac{\alpha + \beta x_n}{A + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (12)$$

(vi)

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (13)$$

(vii)

$$x_{n+1} = \frac{\alpha}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (14)$$

(viii)

$$x_{n+1} = \frac{\beta x_n}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (15)$$

(ix)

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \dots \quad (16)$$

The change of variables

$$x_n = \frac{\sqrt{\alpha}}{y_n}$$

reduces equation (8) to the difference equation

$$y_{n+1} = \frac{B}{y_n} + \frac{C}{y_{n-1}}, \quad n = 0, 1, \dots \quad (17)$$

This equation was investigated in [17] where it was shown that its equilibrium

$$\bar{y} = \sqrt{B + C}$$

is globally asymptotically stable.

Equation (9) is the well-known *Pielou's equation* which was investigated in [11] where it was shown that when

$$\beta \leq A,$$

the zero equilibrium is globally asymptotically stable, and when $x_0 > 0$ and

$$\beta > A,$$

the positive equilibrium

$$\bar{x} = \frac{\beta - A}{C}$$

is globally asymptotically stable. See also [5].

The change of variables

$$x_n = \frac{\beta}{B + Cy_n}$$

reduces equation (10) to the difference equation

$$y_{n+1} = \frac{B + Cy_n}{B + Cy_{n-1}}, \quad n = 0, 1, \dots,$$

which was investigated in [6, Theorem 3.6.3] where it was shown that the positive equilibrium

$$\bar{y} = 1$$

is globally asymptotically stable. See also [5].

Equation (11) is the well-known *Lyness' equation*; see [5], for which it is known that every solution is bounded and persists. Also every solution of equation (1) is periodic with period 5, if and only if

$$\alpha C = \beta^2.$$

Finally, it was shown in [7] and [18] that the equilibrium of equation (11) is locally stable but not locally asymptotically stable.

Equation (12) was investigated in [5], see also [6], where the following results were established.

- (i) Every solution of equation (12) is bounded and persists.
- (ii) The positive equilibrium of equation (12) is globally asymptotically stable when

$$\beta < A,$$

and also when

$$\beta \geq A$$

and

$$\text{either } \alpha C \leq \beta A \text{ or } \beta A < \alpha C \leq 2 \left(\frac{\beta}{A} + 1 \right).$$

Some other special cases where the equilibrium of equation (12) is globally asymptotically stable were discussed in [4].

We believe that the equilibrium of equation (12) is globally asymptotically stable for all positive values of the parameters, and so we offer the following conjecture.

CONJECTURE 1. *The positive equilibrium of equation (12) is globally asymptotically stable.*

Equation (13) was investigated in [2] where it was shown that all solutions are bounded and that the positive equilibrium is globally asymptotically stable when

$$C\beta \leq 4\alpha B + \beta^2.$$

We also believe that the equilibrium of equation (13) is globally asymptotically stable for all positive values of the parameters, and so we offer the following conjecture.

CONJECTURE 2. *The positive equilibrium of equation (13) is globally asymptotically stable.*

By applying linearized stability and Theorem E, we can easily see that positive equilibrium of equation (14) is globally asymptotically stable.

Equation (15) was investigated in [5] where it was shown that the zero equilibrium is globally asymptotically stable if

$$\beta \leq A$$

and that when $x_0 > 0$, the positive equilibrium is globally asymptotically stable if

$$\beta > A.$$

Concerning the *boundedness character* of solutions (see Section 4), among the 21 special cases of equation (1) there are *only two* that may possess *unbounded solutions*. These are the linear equations

$$\begin{aligned} x_{n+1} &= \frac{\beta}{A} x_n, & n &= 0, 1, \dots, \\ x_{n+1} &= \frac{\alpha + \beta x_n}{A}, & n &= 0, 1, \dots, \end{aligned}$$

with

$$\beta > A.$$

Concerning the *periodic nature* of solutions of equation (1), there are only three equations with the property that *every solution is periodic with the same period* $p \geq 2$, namely,

$$x_{n+1} = \frac{\alpha}{Bx_n}, \quad n = 0, 1, \dots \quad (\text{every solution is periodic with period } 2),$$

$$x_{n+1} = \frac{\alpha}{Cx_{n-1}}, \quad n = 0, 1, \dots \quad (\text{every solution is periodic with period } 4),$$

$$x_{n+1} = \frac{\beta x_n}{Cx_{n-1}}, \quad n = 0, 1, \dots \quad (\text{every solution is periodic with period } 6).$$

The special case of equation (1),

$$x_{n+1} = \frac{\alpha + \beta x_n}{Cx_{n-1}}, \quad n = 0, 1, \dots,$$

called *Lyness' equation*, has the property that *every solution is periodic with period 5* if and only if

$$\alpha C = \beta^2.$$

Zero is an equilibrium of equation (1) if and only if

$$\alpha = 0 \quad \text{and} \quad A > 0.$$

In this case, zero is globally asymptotically stable if

$$\beta \leq A,$$

and is unstable if

$$\beta > A.$$

Furthermore, the zero equilibrium is a sink when

$$\beta < A,$$

and a saddle point when $B + C > 0$ and

$$\beta > A$$

(although all solutions with positive initial conditions converge to the positive equilibrium $\bar{x} = (\beta - A)/(B + C)$). In view of the above discussion, there remains to investigate equation (1) and the nature of its positive equilibrium when all the parameters of the equation are positive. In this case, the change of variables

$$x_n = \frac{A}{B} y_n$$

reduces equation (1) to the difference equation

$$y_{n+1} = \frac{p + qy_n}{1 + y_n + ry_{n-1}}, \quad n = 0, 1, \dots, \quad (18)$$

where

$$p = \frac{\alpha B}{A^2}, \quad q = \frac{\beta}{A}, \quad \text{and} \quad r = \frac{C}{B}.$$

In the remaining sections, we will investigate the global character of solutions of equation (18) for various ranges of the parameters

$$p, q, r \in (0, \infty)$$

and with nonnegative initial conditions.

3. LINEARIZED STABILITY

The only equilibrium of equation (18) is

$$\bar{y} = \frac{q-1 + \sqrt{(q-1)^2 + 4p(r+1)}}{2(r+1)},$$

and the linearized equation of equation (18) about \bar{y} is

$$z_{n+1} - \frac{q-p+qr\bar{y}}{(1+(1+r)\bar{y})^2} z_n + \frac{r(p+q\bar{y})}{(1+(1+r)\bar{y})^2} z_{n-1} = 0, \quad n = 0, 1, \dots$$

By employing the linearized stability Theorem A, we obtain the following result.

THEOREM 1. *The equilibrium \bar{y} of equation (18) is locally asymptotically stable for all values of the parameters p , q , and r .*

4. BOUNDEDNESS OF SOLUTIONS

Here, we will prove that all solutions of equation (18) are bounded.

THEOREM 2. *Every solution of equation (18) is bounded from above and from below by positive constants.*

PROOF. Let $\{y_n\}$ be a solution of equation (18). Clearly, if the solution is bounded from above by a constant M , then

$$y_{n+1} \geq \frac{p}{1+(1+r)M},$$

and so it is also bounded from below. Now, assume for the sake of contradiction that the solution is not bounded from above. Then there exists a subsequence $\{y_{1+n_k}\}_{k=0}^{\infty}$ such that

$$\lim n_k = \infty, \quad \lim y_{1+n_k} = \infty, \quad \text{and} \quad y_{1+n_k} = \max\{y_n : n \leq n_k\}, \quad \text{for } k \geq 0.$$

From (18), we see that

$$y_{n+1} < qy_n + p, \quad \text{for } n \geq 0,$$

and so,

$$\lim y_{n_k} = \lim y_{n_k-1} = \infty.$$

Hence, for sufficiently large k ,

$$0 < y_{1+n_k} - y_{n_k} = \frac{p + [(q-1) - y_{n_k} - ry_{n_k-1}]y_{n_k}}{1 + y_{n_k} + ry_{n_k-1}} < 0,$$

which is a contradiction and the proof is complete. ■

The above proof has the advantage that extends to several equations of the form of equation (1) with nonnegative parameters. The boundedness of solutions of the special equation (18) with positive parameters follows from the observation that if

$$M = \max\{1, p, q\},$$

then

$$y_{n+1} \leq \frac{M + My_n}{1 + y_n} = M, \quad \text{for } n \geq 0.$$

5. INVARIANT INTERVALS

We say that the interval I is an *invariant interval* for equation (18) if

$$y_N, y_{N+1} \in I, \quad \text{for some integer } N \geq 0,$$

then

$$y_n \in I, \quad \text{for } n \geq N.$$

LEMMA 1. Equation (18) possesses the following invariant intervals:

(a)

$$\left[0, \frac{q-1 + \sqrt{(q-1)^2 + 4p}}{2} \right], \quad \text{when } p \leq q;$$

(b)

$$\left[\frac{p-q}{qr}, q \right], \quad \text{when } q < p < q(rq+1);$$

(c)

$$\left[q, \frac{p-q}{qr} \right], \quad \text{when } p > q(rq+1).$$

PROOF.

(a) Set

$$g(x) = \frac{p+qx}{1+x} \quad \text{and} \quad b = \frac{q-1 + \sqrt{(q-1)^2 + 4p}}{2}$$

and observe that g is an increasing function and $g(b) \leq b$. Using equation (18), we see that when $y_{k-1}, y_k \in [0, b]$, then

$$y_{k+1} = \frac{p+qy_k}{1+y_k+ry_{k-1}} \leq \frac{p+qy_k}{1+y_k} = g(y_k) \leq g(b) \leq b.$$

The proof follows by induction.

(b) It is clear that the function

$$f(x, y) = \frac{p+qx}{1+x+ry}$$

is increasing in x for $y > (p-q)/qr$. Using equation (18), we see that when $y_{k-1}, y_k \in [(p-q)/qr, q]$, then

$$y_{k+1} = \frac{p+qy_k}{1+y_k+ry_{k-1}} = f(y_k, y_{k-1}) \leq f\left(q, \frac{p-q}{qr}\right) = q.$$

Also by using the condition $p < q(rq+1)$, we obtain

$$\begin{aligned} y_{k+1} &= \frac{p+qy_k}{1+y_k+ry_{k-1}} = f(y_k, y_{k-1}) \geq f\left(\frac{p-q}{qr}, q\right) \\ &= \frac{q(pr+p-q)}{(rq)^2 + rq + p - q} > \frac{p-q}{qr}. \end{aligned}$$

The proof follows by the induction.

(c) It is clear that the function

$$f(x, y) = \frac{p+qx}{1+x+ry}$$

is decreasing in x for $y < (p - q)/qr$. Using equation (18), we see that when $y_{k-1}, y_k \in [q, (p - q)/qr]$, then

$$y_{k+1} = \frac{p + qy_k}{1 + y_k + ry_{k-1}} = f(y_k, y_{k-1}) \geq f\left(\frac{p - q}{qr}, \frac{p - q}{qr}\right) = q.$$

Also, by using the condition $p > q(rq + 1)$, we obtain

$$\begin{aligned} y_{k+1} &= \frac{p + qy_k}{1 + y_k + ry_{k-1}} = f(y_k, y_{k-1}) \leq f(q, q) \\ &= \frac{p + q^2}{1 + (r + 1)q} < \frac{p - q}{qr}. \end{aligned}$$

The proof follows by induction. ■

6. SOME USEFUL IDENTITIES

Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (18). Then the following identities are true for $n \geq 0$:

$$y_{n+1} - q = \frac{qr((p - q)/qr - y_{n-1})}{1 + y_n + ry_{n-1}}, \quad (19)$$

$$y_{n+1} - \frac{p - q}{qr} = \frac{qr(q - (p - q)/qr)y_n + qr(y_{n-1} - (p - q)/qr) + pr(q - y_{n-1})}{qr(1 + y_n + ry_{n-1})}, \quad (20)$$

$$\begin{aligned} y_n - y_{n+4} &= [qr(y_n - (p - q)/qr)y_{n+1} + (y_n - q)(y_{n+1}y_{n+3} + y_{n+3} + y_{n+1} + ry_ny_{n+3}) \\ &\quad + y_n + ry_n^2 - p] \times \frac{1}{(1 + y_{n+3})(1 + y_{n+1} + ry_n) + r(p + qy_{n+1})}, \end{aligned} \quad (21)$$

$$y_{n+1} - \bar{y} = \frac{(\bar{y} - q)(\bar{y} - y_n) + \bar{y}r(\bar{y} - y_{n-1})}{1 + y_n + ry_{n-1}}. \quad (22)$$

The proofs of the following two lemmas are straightforward consequences of identities (19)–(22) and will be omitted.

LEMMA 2. Assume

$$p > q(qr + 1)$$

and let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (18). Then the following statements are true.

- (i) If for some $N \geq 0$, $y_N > (p - q)/qr$, then $y_{N+2} < q$.
- (ii) If for some $N \geq 0$, $y_N = (p - q)/qr$, then $y_{N+2} = q$.
- (iii) If for some $N \geq 0$, $y_N < (p - q)/qr$, then $y_{N+2} > q$.
- (iv) If for some $N \geq 0$, $q < y_N < (p - q)/qr$, then $q < y_{N+2} < (p - q)/qr$.
- (v) If for some $N \geq 0$, $\bar{y} \geq y_{N-1}$ and $\bar{y} \geq y_N$, then $y_{N+1} \geq \bar{y}$.
- (vi) If for some $N \geq 0$, $\bar{y} < y_{N-1}$ and $\bar{y} < y_N$, then $y_{N+1} < \bar{y}$.
- (vii) $q < \bar{y} < (p - q)/qr$.

LEMMA 3. Assume

$$q < p < q(qr + 1)$$

and let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (18). Then the following statements are true.

- (i) If for some $N \geq 0$, $y_N < (p - q)/qr$, then $y_{N+2} > q$.
- (ii) If for some $N \geq 0$, $y_N = (p - q)/qr$, then $y_{N+2} = q$.
- (iii) If for some $N \geq 0$, $y_N > (p - q)/qr$, then $y_{N+2} < q$.
- (iv) If for some $N \geq 0$, $y_N > (p - q)/qr$ and $y_N < q$ then $q > y_{N+2} > (p - q)/qr$.
- (v) $(p - q)/qr < \bar{y} < q$.

LEMMA 4. Assume

$$p = q(qr + 1)$$

and let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (18). Then

$$y_{n+1} - q = \frac{qr}{1 + y_n + ry_{n-1}}(q - y_{n-1}). \quad (23)$$

Furthermore, if $qr < 1$, then

$$\lim_{n \rightarrow \infty} y_n = \bar{y}. \quad (24)$$

PROOF. Identity (23) follows by straightforward computation. Limit (24) is a consequence of the fact that in this case $qr \in (0, 1)$ and equation (18) has no prime period two solution. ■

7. GLOBAL ASYMPTOTIC STABILITY

The following lemma establishes that when $p \neq q(qr + 1)$, every solution of equation (18) is eventually trapped into one of the three invariant intervals of equation (18) described in Lemma 1. More precisely, the following is true.

LEMMA 5. Let I denote the interval which is defined as follows:

$$I = \begin{cases} \left[0, \frac{q-1 + \sqrt{(q-1)^2 + 4p}}{2} \right], & \text{if } p \leq q, \\ \left[\frac{p-q}{qr}, q \right], & \text{if } q < p < q(qr + 1), \\ \left[q, \frac{p-q}{qr} \right], & \text{if } p > q(qr + 1). \end{cases}$$

Then every solution of equation (18) lies eventually in I .

PROOF. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of equation (18). First assume that

$$p \leq q.$$

Then, clearly, $\bar{y} \in I$. Set

$$b = \frac{q-1 + \sqrt{(q-1)^2 + 4p}}{2}$$

and observe that

$$y_{n+1} - b = \frac{(q-p)(y_n - b) - r(p+qb)y_{n-1}}{(1+y_n+ry_{n-1})(1+b)}, \quad n \geq 0.$$

Hence, if for some N , $y_N \leq b$, then $y_{N+1} < b$. Now, assume for the sake of contradiction that

$$y_n > b, \quad \text{for } n > 0.$$

Then, $y_n > \bar{y}$ for $n \geq 0$ and so clearly

$$\lim_{n \rightarrow \infty} y_n = \bar{y} \in I,$$

which is a contradiction.

Next, assume that

$$q < p < q(qr + 1)$$

and for the sake of contradiction also assume that the solution $\{y_n\}_{n=-1}^{\infty}$ is not eventually in the interval I . Then, by Lemma 3, there exists $N > 0$ such that one of the following three cases holds:

- (i) $y_N > q$, $y_{N+1} > q$, and $y_{N+2} < (p-q)/qr$;
- (ii) $y_N > q$, $y_{N+1} < (p-q)/qr$, and $y_{N+2} < (p-q)/qr$;
- (iii) $y_N > q$, $(p-q)/qr \leq y_{N+1} \leq q$, and $y_{N+2} < (p-q)/qr$.

Also, observe that if $y_N \geq q$, then $y_N + ry_N^2 - p > 0$ and if $y_N \leq (p-q)/qr$, then $y_N + ry_N^2 - p < 0$.

The desired contradiction is now obtained by using the identity (21) from which it follows that for $j \in \{0, 1, 2, 3\}$, each subsequence $\{y_{n+4k+j}\}_{k=0}^{\infty}$ with all its terms outside the interval I converges monotonically, and so it must enter in the interval I .

The proof when

$$p > q(qr + 1)$$

is similar and will be omitted. ■

By using the monotonic character of the function

$$f(x, y) = \frac{p + qx}{1 + x + ry}$$

in each of the intervals in Lemma 1, together with the appropriate convergence Theorems B and C, we can obtain some global asymptotic stability results for the solutions of equation (18). For example, the following results are true for equation (18).

THEOREM 3.

(a) Assume that either

$$p \geq q + q^2r$$

or

$$p < \frac{q}{1+r}.$$

Then the equilibrium \bar{y} of equation (18) is globally asymptotically stable.

(b) Assume that either

$$p \leq q$$

or

$$q < p < q + q^2r$$

and that one of the following conditions is also satisfied:

- (i) $q \leq 1$;
- (ii) $r \leq 1$;
- (iii) $r > 1$ and $(q-1)^2(r-1) \leq 4p$.

Then the equilibrium \bar{y} of equation (18) is globally asymptotically stable.

PROOF.

(a) The proof follows from Lemmas 1 and 4 and Theorems C and D.

(b) In view of Lemma 1, we see that when $y_{-1}, y_0 \in [0, (q-1 + \sqrt{(q-1)^2 + 4p})/2]$, then $y_n \in [0, (q-1 + \sqrt{(q-1)^2 + 4p})/2]$ for all $n \geq 0$. It is easy to check that $\bar{y} \in [0, (q-1 + \sqrt{(q-1)^2 + 4p})/2]$ and that in the interval $[0, (q-1 + \sqrt{(q-1)^2 + 4p})/2]$, the function f increases in x and decreases in y .

We will employ Theorem B, and so it remains to show that if

$$m = f(m, M) \quad \text{and} \quad M = f(M, m),$$

then $M = m$. This system has the form

$$m = \frac{p + qm}{1 + m + rM} \quad \text{and} \quad M = \frac{p + qM}{1 + M + rm}.$$

Hence, $(M - m)(1 - q + M + m) = 0$. Now if $m + M \neq q - 1$, then $M = m$. For instance, this is the case if Condition (i) is satisfied. If $m + M = q - 1$, then m and M satisfy the equation

$$(r-1)m^2 + (r-1)(1-q)m + p = 0.$$

Clearly now, if Condition (ii) or (iii) is satisfied, $m = M$ from which the result follows.

The proof when $q < p < q + q^2r$ holds follows from Lemma 1 and Theorem B. ■

We believe that the positive equilibrium of equation (18) is globally asymptotically stable for all values of the parameters and so we offer the following conjecture.

CONJECTURE 3. *The positive equilibrium of equation (18) is globally asymptotically stable.*

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